Nambu Bracket for M Theory

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Generalized Hamiltonian Dynamics*

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Taking the Liouville theorem as a guiding principle, we propose a possible generalization of classical Hamiltonian dynamics to a three-dimensional phase space. The equation of motion involves two Hamiltonians and three canonical variables. The fact that the Euler equations for a rotator can be cast into this form suggests the potential usefulness of the formalism. In this article we study its general properties and the problem of quantization.

I. INTRODUCTION

A notable feature of the Hamiltonian description of classical dynamics is the Liouville theorem, which states that the volume of phase space occupied by an ensemble of systems is conserved. The theorem plays, among other things, a fundamental role in statistical mechanics. On the other hand, Hamiltonian dynamics is not the only formalism that makes a statistical mechanics possible. Any set of equations which lead to a Liouville theorem in a suitably defined phase space will do (provided of course that ergodicity may be assumed). With this in mind, let us consider the following scheme.

Let $(x, y, z) \equiv \vec{r}$ be a triplet of dynamical variables (canonical triplet) which span a three-dimensional phase space. This is a formal generalization of the conventional phase space spanned by a canonical pair (p, q). Next introduce two functions, H and

$[F, H, G]$. Obviously a PB is antisymmetric under interchange of any pair of its components. As a result we have $H = F = 0$, i.e., both H and G are constants of motion. The orbit of a system in phase space is thus determined as the intersection of two surfaces, $H = \text{const.}$ and $G = \text{const.}$

Equation (1) or (1′) also shows that the velocity field $d\vec{r}/dt$ is divergenceless,

$$\vec{\nabla}\cdot(\vec{\nabla}H \times \vec{\nabla}G) \equiv 0, \tag{3}$$

and this amounts to a Liouville theorem in our phase space.

The above properties immediately tempt us to construct a statistical mechanics where a canonical ensemble is characterized by a generalized Boltzmann distribution in phase space with a weight factor

$$e^{-\beta H - \gamma G}.$$

Nambu-Poisson bracket

- Generalization of Poisson brackets [Takhtajan 94]
- $\{f, g, h\} = P^{abc}(x)\, \partial_a f(x)\, \partial_b g(x)\, \partial_c h(x)$
- Example: $(x^1, x^2, x^3) \in \mathbb{R}^3$

$$\{f, g, h\} = \varepsilon^{abc}(x)\, \partial_a f(x)\, \partial_b g(x)\, \partial_c h(x)$$

$$(a, b, c = 1,2,3)$$

1. Skew-symmetry
2. Fundamental identity (generalized Jacobi id.)
3. Leibniz rule (not required for Lie 3-algebra)

$$\{f, g, h_1.h_2\} = \{f, g, h_1\}h_2 + \{f, g, h_2\}h_1$$

Fundamental Identity

Symmetry transformation

$$\delta_{(A,B)}C = \{A, B, C\}$$

Closure:

$$[\delta_{(A,B)}, \delta_{(C,D)}] = \delta_{(E,F)} + \delta_{(G,H)}$$

$(E, F) = (\delta_{(A,B)}C, D),$ $(G, H) = (C, \delta_{(A,B)}D)$

Covariance:

$$\delta_{(A,B)}\{C, D, E\} = \{\delta_{(A,B)}C, D, E\} + \{C, \delta_{(A,B)}D, E\} + \{C, D, \delta_{(A,B)}E\}$$

- For Poisson bracket,

  Darboux' theorem:

  Locally, P^{ab} can be described in terms of canonical variables,

$$\{p_i , q_j\} = \delta_{ij}$$

- For Nambu-Poisson bracket,

  Decomposability theorem:

  Locally, $P^{abc} = \varepsilon^{abc}$ for 3 of the n coordinates,

$$\{x^a, x^b, x^c\} = \varepsilon^{abc}$$

Volume-Preserving Diffeomorphism

- NP bracket generates volume-preserving diffeomorphisms.

  generalization of Liouville theorem

- Example: $\{ x^a, x^b, x^c \} = \varepsilon^{abc}$

  $\delta_\Lambda f(x) = \sum_{ij} \Lambda_{ij} \{\phi^i, \phi^j, f\} = \varepsilon^{abc}\, \partial_a \Lambda_b(x)\, \partial_c f(x)$

$$\Lambda_b(x) \equiv \sum_{ij} \Lambda_{ij}\, \phi^i(x)\, \partial_b \phi^j(x)$$

$$\Rightarrow \delta_\Lambda (dx^1 dx^2 dx^3) = 0$$

gauge theory?

M5-brane in M theory

- Self-dual 3-form gauge field in 6 dim's.

  Pasti-Sorokin-Tonin [1997]

  Ho-Matsuo 0804.3629 [2008]

  Ho-Imamura-Matsuo-Shiba 0805.2898 [2008]

- In a constant C-field (3-form) backgroud

$$x = (x^0, x^1, x^2) \in \mathbb{R}^{1+2}$$

$$y = (x^3, x^4, x^5) \in \mathbb{R}^3 \leftarrow C$$

- 2-form gauge potential:

$$A_{xx}, \boldsymbol{A_{xy}}, \boldsymbol{A_{yy}}$$

$$F_{xxx}, F_{xxy}, \boldsymbol{F_{xyy}}, \boldsymbol{F_{yyy}}$$

Linearized Self-Dual 3-Form Gauge Field

Gauge transformation:

$$\begin{aligned}\delta A_{\dot\mu\dot\nu}(x,y) &= \partial_{\dot\mu}\Lambda_{\dot\nu}(x,y) - \partial_{\dot\nu}\Lambda_{\dot\mu}(x,y),\\ \delta A_{\mu\dot\mu}(x,y) &= \partial_{\mu}\Lambda_{\dot\mu}(x,y).\end{aligned}$$

Equations of motion:

$$\partial_{\underline{\mu}} F_{\underline{\mu}\dot\mu\dot\nu} = 0,$$

$$\partial_{\dot\nu} F_{\dot\nu\mu\dot\mu} + \partial_{\nu}\tilde{F}_{\nu\mu\dot\mu} = 0. \qquad \Rightarrow \qquad F_{\mu\dot\mu\dot\nu} - \tilde{F}_{\mu\dot\mu\dot\nu} = \epsilon_{\dot\mu\dot\nu\dot\lambda}\partial_{\dot\lambda}B_{\mu}.$$

2-form gauge potential:

$$B_{\mu\nu} = -\epsilon_{\mu\nu\lambda}B_{\lambda}, \qquad B_{\mu\dot\mu} = A_{\mu\dot\mu}, \qquad B_{\dot\mu\dot\nu} = A_{\dot\mu\dot\nu}.$$

Self-dual conditions:

$$H_{\mu\nu\lambda} = \frac{1}{6}\epsilon_{\mu\nu\lambda}\epsilon^{\dot\mu\dot\nu\dot\lambda}H_{\dot\mu\dot\nu\dot\lambda}, \qquad H_{\dot\mu\dot\nu\dot\lambda} = \frac{1}{6}\epsilon_{\mu\nu\lambda}\epsilon^{\dot\mu\dot\nu\dot\lambda}H_{\mu\nu\lambda},$$

$$H_{\mu\nu\dot\mu} = -\frac{1}{2}\epsilon_{\mu\nu\lambda}\epsilon^{\dot\mu\dot\nu\dot\lambda}H_{\lambda\dot\nu\dot\lambda}, \qquad H_{\mu\dot\mu\dot\nu} = \frac{1}{2}\epsilon_{\mu\nu\lambda}\epsilon^{\dot\mu\dot\nu\dot\lambda}H_{\nu\lambda\dot\lambda}.$$

M5-brane in C-field background

$$
\begin{aligned}
\mathcal{D}_{\dot\mu}\Phi &\equiv \frac{g^2}{2}\epsilon_{\dot\mu\dot\nu\dot\rho}\{X^{\dot\nu}, X^{\dot\rho}, \Phi\} \\
&= \partial_{\dot\mu}\Phi + g(\partial_{\dot\lambda} b^{\dot\lambda}\partial_{\dot\mu}\Phi - \partial_{\dot\mu} b^{\dot\lambda}\partial_{\dot\lambda}\Phi) + \frac{g^2}{2}\epsilon_{\dot\mu\dot\nu\dot\rho}\{b^{\dot\nu}, b^{\dot\rho}, \Phi\}.
\end{aligned}
$$

$$
\mathcal{D}_{\mu}\Phi \equiv D_\mu\Phi = \partial_\mu\Phi - g\{b_{\mu\dot\nu}, y^{\dot\nu}, \Phi\}
$$

$$
\begin{aligned}
S_X + S_{\text{pot}} &= \int d^3x \left\langle -\frac{1}{2}(\mathcal{D}_\mu X^i)^2 - \frac{1}{2}(\mathcal{D}_{\dot\lambda} X^i)^2 - \frac{1}{4}\mathcal{H}^2_{\lambda\dot\mu\dot\nu} - \frac{1}{12}\mathcal{H}^2_{\dot\mu\dot\nu\dot\rho} \right. \\
&\left. -\frac{1}{2g^2} - \frac{g^4}{4}\{X^{\dot\mu}, X^i, X^j\}^2 - \frac{g^4}{12}\{X^i, X^j, X^k\}^2 \right\rangle, \\
S_\Psi + S_{\text{int}} &= \int d^3x \left\langle \frac{i}{2}\overline{\Psi}\Gamma^\mu\mathcal{D}_\mu\Psi + \frac{i}{2}\overline{\Psi}\Gamma^{\dot\rho}\Gamma_{\dot1\dot2\dot3}\mathcal{D}_{\dot\rho}\Psi \right. \\
&\left. +\frac{ig^2}{2}\overline{\Psi}\Gamma_{\dot\mu i}\{X^{\dot\mu}, X^i, \Psi\} + \frac{ig^2}{4}\overline{\Psi}\Gamma_{ij}\{X^i, X^j, \Psi\} \right\rangle.
\end{aligned}
$$

Complete nonlinear action of M5-brane with self-dual gauge fields in C-field is derived from Bagger-Lambert-Gustavsson model [2007, 2008] for multiple M2-branes with Lie 3-algebra = NP bracket:

- Supersymmetry
- Gauge symmetry defined by NP bracket
- Self-dual 3-form gauge field
- Seiberg-Witten map

Conclusion

- Nambu-Poisson bracket ↔

  gauge symmetry of volume-preserving diffeomorphism
- Self-dual 3-form gauge field theory on 6D

Future directions

- Quantization of Nambu-Poisson bracket

  Lie 3-algebra, nonassociative geometry,…
- More understanding and more applications